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Polynomial-like semi-conjugates of the shift map. *

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Abstract

In this paper I prove that for a polynomial of degree d with a Cantor Julia set J, the Julia set can be understood as the simplest possible quotiont of the one sided shift space Σ_d with dynamics given by the shift. Here simplest possible means that, the projection $\pi : \Sigma_d \longrightarrow J$ is as injective as possible.

1 Introduction

Denote by $\Sigma_d = \{0, \ldots, d-1\}^{\mathbb{N}}$ the set of one-sided infinite sequences of symbols the $0, \ldots, d-1$ equipped with the natural product topology. And denote by $\sigma : \Sigma_d \longrightarrow \Sigma_d$ the shift map:

$$\sigma((\epsilon_i)_i) = (\epsilon_{i+1})_i = (\epsilon_2, \epsilon_3, \ldots).$$

Douady and Hubbard introduced in [DH] the notion of polynomial-like maps. Here we shall use a slightly generalized version of such maps (see also [L-V]):

Let $f : U' \longrightarrow U$ be a proper holomorphic map where $U \simeq \mathbb{D}$, $U' \subset\subset U$, $U' = U_1 \cup U_1 \cup \ldots \cup U_N$, $U_i \simeq \mathbb{D}$ for each i and $U_i \cap U_j = \emptyset$ for $i \neq j$.

The filled-in Julia set K_f for f is the set of points:

$$K_f = \{z \in U' | f^n(z) \in U', \forall\, n \in \mathbb{N}\},$$

and the Julia set is its topological boundary $J_f = \partial K_f$.

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For such a map the degree d is the sum of the degrees d_i of the restrictions $f_| : U_i \longrightarrow U$. By the Riemann-Hurwitz formula f has counting multiplicity $d'_i = d_i - 1$ critical points in U_i. In particular if f does not have any critical point in some U_i, then f has a globally defined inverse branch $f_i = f_|^{-1} : U \longrightarrow U_i$. In particular if f has no critical points at all then $d = N$ and f has d distinct globally defined inverse branches. In this case it follows that $K_f = J_f$ is a Cantor set and an elementary proof going back to Fatou shows that in the later case there is a homeomorphism $\pi : \Sigma_d \longrightarrow J_f$ such that $\pi \circ \sigma = f \circ \pi$.

If no critical point of f is periodic then the function $\chi : K_f \longrightarrow \mathbb{N}$ given by the maximal local degree of iterates of f near z:

$$\chi(z) = \sup_{n\in\mathbb{N}} \deg(f^n, z)$$

is bounded by the product of the local degrees of f at its critical points and satisfies

$$\chi(z) = \deg(f, z) \cdot \chi(f(z)).$$

Since f has $d' = d - N = \sum d'_i$ critical points the function χ is bounded by $2^{d'}$. Note that any periodic critical point is surrounded by an open attracted basin, and thus belongs to the interior of K_f.

The main theorem of this paper is:

Theorem 1.1. *Let $f : U' \longrightarrow U$ be a degree $d > 1$ generalized polynomial-like map in the sense above. If $K_f = J_f$ is a Cantor set containing all critical points of f. Then there is a semi-conjugacy $\pi : \Sigma_d \longrightarrow J_f$, $\pi \circ \sigma = f \circ \pi$ such that:*

$$\forall z \in J_f : \#\pi^{-1}(z) = \chi(z).$$

Remark 1) Branner and Hubbard proved in [B-H] that there are many cubic polynomials with a generalized polynomial-like restriction as above satisfying the hypothesis and thus the conclusion of the above Theorem. Moreover recently this Branner-Hubbard Theorem has been extend to all degrees and all orders of critical points. See e.g. [T-Y], [Y] and [K-S].

Remark 2) The main theorem is related to the structure of the complement of the cubic connectedness locus through ther paper [DGK] of Devaney, Goldberg and Keen.

Remark 3) The injectivity statement of the main Theorem is best possible, since if for some point z :$\#\pi^{-1}(P(z)) = l$ and if the local degree of f at z is $m \geq 1$. Then $\#\pi^{-1}(z) = ml$.

Remark 4) $N \geq 2$ since if not J_f would be connected and not a Cantor set.

Remark 5) The hypothesis that J_f is Cantor set is equivalent to asking that

the diameters of the connected components of $f^{-n}(U)$ converge to zero as n tends to infinity.

Towards a proof of the main theorem we introduce some notation.
We shall in the following tacitly assume the hypothesies of the Theorem, i.e. $f: U' \longrightarrow U$ *is a generalized polynomial-like map for which* $K_f = J_f$ *is a Cantor set containing all critical points of* f.
Note that taking a restriction with U slightly smaller if necessary we can assume the boundaries of all disks U and U_i are smooth and disjoint. Let $w \in U \backslash U'$ be arbitrary and let $w_0, \ldots, w_{d-1}$ denote the d distinct preimages of w, and let $i = i(j)$ denote the function given by $w_j \in U_{i(j)}$. Renumbering if necessary we can assume that i is weakly increasing, i.e. we have filled-in from below.

Let $\phi : \mathbb{D} \longrightarrow U \backslash J_f$ be a universal covering with $\phi(0) = w$.

Proposition 1.2. *There exist* d *(univalent) lifts* $g_i : \mathbb{D} \longrightarrow \mathbb{D}$, $i = 0, \ldots, d-1$ *of* ϕ *to* $f \circ \phi$, *i.e.* $f \circ \phi \circ g_i = \phi$ *with* $\phi \circ g_i(0) = w_i$. *These satisfy* $\phi \circ g_j(z) \neq \phi \circ g_{j'}(z)$ *for* $j \neq j' \mod d$, *i.e. for any* $z \in \mathbb{D}$ *the points* $\phi \circ g_j(z)$ *are the* d *distinct preimages of* $\phi(z)$ *under* f. *In particular*

$$f^{-1}(\phi(\mathbb{D})) = f^{-1}(U \backslash J_f) = U' \backslash J_f = \bigcup_{i=0}^{d-1} (\phi \circ g_i)(\mathbb{D}).$$

Remark that the g_i are by no means unique.

Proof. For $0 \leq j < d$ and $i = i(j)$ let V_i be a connected component of $\phi^{-1}(U_i \backslash J_f)$. Then V_i is simply connected, because $U_i \backslash J_f$ is a retract of $U \backslash J_f$. Hence the restriction $\phi : V_i \longrightarrow U_i \backslash J_f$ is a universal covering map. Since the restriction $f : U_i \backslash J_f \longrightarrow U \backslash J_f$ has no critical points it is also a covering and thus each $f \circ \phi_{|V_i}$ is a universal covering. Let $x_j \in V_i$ be any point with $\phi(x_j) = w_j$. Then there is a unique lift $g_j : \mathbb{D} \longrightarrow V_j$ of the universal covering ϕ to the (universal) covering $f \circ \phi_{|V_j}$ mapping 0 to x_j. Being lifts of $f \circ \phi$ to ϕ, any two of the g_j either egree everywhere or nowhere. They are chosen to disagree at 0. $\square$

Note that changing the choice of some z_j to some other preimage z_j' of w_j amounts to post composing g_j with the decktransformation for ϕ, which maps z_j to z_j'. We shall think and speak of the maps g_j as lifts of f^{-1} though technically they are self-maps of a different space.

For $k \geq 1$ let Σ_d^k denote the set of k-blocks $\epsilon^k = (\epsilon_1, \ldots, \epsilon_k)$ in the alphabet $\{0, \ldots, d-1\}$. Every such ϵ^k defines a "cylinder" clopen set

$$\{(\tau_j)_j | \tau_j = \epsilon_j, j \leq k\}.$$

The map σ thus has a natural extension as a map from Σ_d^k to Σ_d^{k-1}. Also for $n < k$ there is a natural projection from Σ_d^k to Σ_d^n: $\epsilon^k \to \epsilon^n$, which simply forgets the last $k-n$ entries.

The obvious idea for proving Theorem 1.1 would now be to iterate the d branches $g_j : \mathbb{D} \longrightarrow \mathbb{D}$ of the inverse of f, project back to U and obtain sets for defining a semiconjugacy. More precisely for $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ldots) \in \Sigma_d$ and $k \in \mathbb{N}$ define

$$g_{\epsilon^k} = g_{\epsilon_1} \circ \ldots \circ g_{\epsilon_k},$$

and

$$V_{\epsilon^k} = g_{\epsilon^k}(\mathbb{D}).$$

Then for each $\underline{\epsilon} \in \Sigma_d$ the set $\cap_{k \geq 1} \phi(V_{\epsilon^k})$ is a connected subset of the Cantor set J_f and thus a singleton $\{z_{\underline{\epsilon}}\}$. Define $\Psi : \Sigma_d \longrightarrow J_f$ by $\Psi(\underline{\epsilon}) = z_{\underline{\epsilon}}$. Then Ψ is indeed a semi-conjugacy of $\sigma : \Sigma_d \longrightarrow \Sigma_d$ to $P : J_f \longrightarrow J_f$. However in general it will not have the promised injectivity properties. The problem originates in the number of connected components V_{ϵ^k}, $\epsilon^k \in \Sigma_d^k$ is groving much faster than the number of connected components of $f^{-k}(U)$. To remove this problem we shall use decktransformations for ϕ to push together the sets V_{ϵ^k} and $V_{\widehat{\epsilon}^k}$ whenever $\phi(V_{\epsilon^k}) = \phi(V_{\widehat{\epsilon}^k})$.

To fix the ideas let Γ denote the group of decktransformations for the universal covering ϕ, i.e $\gamma \in \Gamma$, if and only if γ is an automorphism of $\mathbb{D}$ with $\phi \circ \gamma = \phi$.

Lemma 1.3. *Given $\epsilon^k \in \Sigma_d^k$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ let*

$$V = \gamma_1 \circ g_{\epsilon_1} \circ \ldots \circ \gamma_k \circ g_{\epsilon_k}(\mathbb{D}) \quad \text{and} \quad V' = \gamma_2 \circ g_{\epsilon_2} \circ \ldots \circ \gamma_k \circ g_{\epsilon_k}(\mathbb{D}).$$

Then the restrictions $\phi : V \longrightarrow \phi(V)$ and $\phi : V' \longrightarrow \phi(V')$ are universal coverings, $\phi(V) = W \backslash J_f$, $\phi(V') = W' \backslash J_f$, where W, W' are connected components of $f^{-k}(U)$ and $f^{-(k-1)}(U)$ respectively and the restriction $f : W \longrightarrow W'$ is a branched covering.

Proof. The first statements is an easy induction proof, based on Proposition 1.2, the details are left to the reader. The last statement follows from

$$f \circ \phi \circ \gamma_1 \circ g_{\epsilon_1} \circ \gamma_2 \circ \ldots \circ \gamma_k \circ g_{\epsilon_k} = \phi \circ \gamma_2 \circ g_{\epsilon_2} \circ \ldots \gamma_k \circ g_{\epsilon_k}.$$

□

Note also that

$$V \subset \gamma_1 \circ g_{\epsilon_1} \circ \ldots \circ \gamma_{k-1} \circ g_{\epsilon_{(k-1)}}(\mathbb{D}).$$

Proposition 1.4. *There exists a sequence of families* $\{\gamma_l^{\epsilon^k}\}_{l=1}^k \subset \Gamma$, $\epsilon^k \in \Sigma_d^k$, $k \in \mathbb{N}$ *such that the family of sets* $V_{\epsilon^k} := \gamma_1^{\epsilon^k} \circ g_{\epsilon_1} \circ \ldots \circ \gamma_k^{\epsilon^k} \circ g_{\epsilon_k}(\mathbb{D})$ *and the sequence of families of decktransformations* $\{\gamma_l^{\epsilon^k}\}_{l=1}^k$ *satisfies the following three properties:*

1. *For all* $k \geq 2$ *and for all* ϵ^k : $V_{\epsilon^k} \subset V_{\epsilon^{k-1}}$, *where* $\epsilon^{k-1} = \epsilon_1 \ldots \epsilon_{k-1}$.
2. *For all* $k \geq 2$, *for all* ϵ^k *and for all* $l = 2, \ldots, k$: $\gamma_l^{\epsilon^k} = \gamma_{l-1}^{\sigma(\epsilon^k)}$.
3. *For all* $k \geq 1$ *and for all* $\epsilon^k, \widehat{\epsilon}^k$: *If* $\phi(V_{\epsilon^k}) = \phi(V_{\widehat{\epsilon}^k})$, *then* $V_{\epsilon^k} = V_{\widehat{\epsilon}^k}$.

Remark that 3. implies that there is a $1 : 1$ correspondence between connected components of $f^{-k}(U)$ and connected components of $\cup_{\epsilon^k \in \Sigma_d^k} V_{\epsilon^k}$. And that 2. imples that this correspondence agrees with the dynamics, i.e. $(f \circ \phi)(V_{\epsilon^k}) = \phi(V_{\sigma(\epsilon^k)})$.

Proof. The proof is by induction on k. For this it is convenient to let $\emptyset$ denote the empty tuple of length 0 and define $\sigma(\epsilon^1) = \emptyset$. Also we shall then extextend the above properties 1. and 2. to $k = 1$ and property 3. to $k = 0$. We then define $V_\emptyset = \mathbb{D}$. This takes care of $k = 0$. For $k = 1$ we have already chosen the branches g_j of the lifted inverse of f so that $g_j(\mathbb{D}) = g_{j'}(\mathbb{D})$ whenever $\phi(g_j(\mathbb{D})) = \phi(g_{j'}(\mathbb{D}))$. Thus we can simply take each $\gamma_1^{\epsilon^1} = \mathrm{id}$. This then complies with all three properties. For the inductive step suppose families $\{\gamma_l^{\epsilon^n}\}_{l=1}^n \subset \Gamma$, $\epsilon^n \in \Sigma_d^n$, $0 \leq n < k$ satisfying the three properties have been constructed. For any $\epsilon^k \in \Sigma_d^k$ define $\gamma_l^{\epsilon^k} = \gamma_{l-1}^{\sigma(\epsilon^k)}$ for $1 < l \leq k$. Moreover define $\widehat{\gamma}_1^{\epsilon^k} = \gamma_1^{\epsilon^{k-1}}$ and $\widehat{V}_{\epsilon^k} = \widehat{\gamma}_1^{\epsilon^k} \circ g_{\epsilon_1}(V_{\sigma(\epsilon^k)})$ as preliminary candidates for $\gamma_1^{\epsilon^k}$ and V^{ϵ^k}. With this choice 2. is immediately satisfied and hence so is 1., because

$$
\begin{aligned}
V_{\sigma(\epsilon^k)} &= \gamma_1^{\sigma(\epsilon^k)} \circ g_{\epsilon_2} \circ \gamma_2^{\sigma(\epsilon^k)} \circ g_{\epsilon_3} \circ \ldots \circ \gamma_{k-1}^{\sigma(\epsilon^k)} \circ g_{\epsilon_k}(\mathbb{D}) \\
&= \gamma_2^{\epsilon^k} \circ g_{\epsilon_2} \circ \gamma_3^{\epsilon^k} \circ g_{\epsilon_3} \circ \ldots \circ \gamma_k^{\epsilon^k} \circ g_{\epsilon_k}(\mathbb{D})
\end{aligned}
$$

and by the induction hypothesis $V_{\sigma(\epsilon^{k-1})} \supset V_{\sigma(\epsilon^k)}$, so that

$$
\widehat{V}_{\epsilon^k} = \gamma_1^{\epsilon^{k-1}} \circ g_{\epsilon_1}(V_{\sigma(\epsilon^k)}) \subset \gamma_1^{\epsilon^{k-1}} \circ g_{\epsilon_1}(V_{\sigma(\epsilon^{k-1})}) = V_{\epsilon^{k-1}}.
$$

To complete the inductive step suppose $\phi(\widehat{V}_{\epsilon^k}) = \phi(\widehat{V}_{\widehat{\epsilon}^k})$. Then by the above $\widehat{V}_{\epsilon^k} \subset V_{\epsilon^{k-1}}$, $\widehat{V}_{\widehat{\epsilon}^k} \subset V_{\widehat{\epsilon}^{k-1}}$ so that

$$
\phi(V_{\epsilon^{k-1}}) = \phi(V_{\widehat{\epsilon}^{k-1}}).
$$

And thus $V_{\epsilon^{k-1}} = V_{\widehat{\epsilon}^{k-1}}$ by property 3. applied to ϵ^{k-1} and $\widehat{\epsilon}^{k-1}$. That is $\phi(\widehat{V}_{\epsilon^k}) = \phi(\widehat{V}_{\widehat{\epsilon}^k})$ implies

$$\widehat{V}_{\epsilon^k}, \widehat{V}_{\widehat{\epsilon}^k} \subset V_{\epsilon^{k-1}} = V_{\widehat{\epsilon}^{k-1}}.$$

Define an equivalence relation $\sim$ on Σ_d^k by

$$\epsilon^k \sim \widehat{\epsilon}^k \Leftrightarrow \phi(\widehat{V}_{\epsilon^k}) = \phi(\widehat{V}_{\widehat{\epsilon}^k}).$$

For each equivalence class of $\sim$ choose a prefered representative ϵ^k, e.g. the one which is minimal with repsect to the lexicographic ordering, and define $\gamma_1^{\epsilon^k} = \widehat{\gamma}_1^{\epsilon^k}$, $V_{\epsilon^k} = \widehat{V}_{\epsilon^k}$. For any other element $\widehat{\epsilon}^k \in [\epsilon^k]$ choose $\gamma_1^{\widehat{\epsilon}^k} \in \Gamma$ so that

$$V_{\widehat{\epsilon}^k} = \gamma_1^{\widehat{\epsilon}^k} \circ g_{\epsilon_1}(V_{\sigma(\widehat{\epsilon}^k)}) = V_{\epsilon^k}.$$

Then also property 3. is satisfied. $\square$

We have now laid the grounds for the projection $\pi : \Sigma_d \longrightarrow J_f$ of the Main Theorem: Define the projection mapping $\pi = \pi_f : \Sigma_d \longrightarrow J_f$ by

$$\pi((\epsilon_j)_j) = \bigcap_{k=1}^{\infty} \overline{\phi(V_{\epsilon^k})}.$$

Then by construction the map π is continuous and semi-conjugates the shift σ on Σ_d to f on J_f

$$\pi \circ \sigma = f \circ \pi.$$

The rest of the paper is devoted to proving that π is as stated in the theorem: i.e. is surjective, is injective above any non-(pre)critical point z and for any $z \in J_f$ satisfies

$$\#\pi^{-1}(z) = \deg(f, z) \cdot \#\pi^{-1}(f(z)).$$

Let us first address the issue of surjectivity.

Proposition 1.5. *For any $k \in \mathbb{N}$:*

$$f^{-k}(U \backslash J_f) = \bigcup_{\epsilon^k \in \Sigma_d^k} \phi(V_{\epsilon^k})$$

Proof. This is an elementary induction proof based on Proposition 1.2 and the observation that for any j and any decktransformation $\gamma \in \Gamma$: $\phi \circ \gamma \circ g_j = \phi \circ g_j$. Combining the observation with Proposition 1.2 shows that the

statement holds for $k = 1$. Now suppose the statement holds for some k. Then by Proposition 1.2:

$$f^{-(k+1)}(U \backslash J_f) = \bigcup_{j=0}^{d-1} \bigcup_{\epsilon^k \in \Sigma_d^k} \phi \circ g_j(V_{\epsilon^k}) = \bigcup_{\epsilon^{(k+1)} \in \Sigma_d^{(k+1)}} \phi(V_{\epsilon^{(k+1)}})$$

And thus the inductive step follows from the observation.. $\square$

Proposition 1.6. *The map $\pi = \pi_f : \Sigma_d \longrightarrow J_f$ is surjective.*

Proof. Let $z \in J_f$ be arbitrary and let W_k denote the connected component of $f^{-k}(U)$ containing z. Then by Proposition 1.5, there exists a sequence $(\epsilon^k)_k$, $\epsilon^k \in \Sigma_d^k$ for each k such that $\phi(V_{\epsilon^k}) = W_k \backslash J_f$. By compactness of Σ_d there exists at least one accumulation point $\underline{\epsilon} \in \Sigma_d$ of $(\epsilon^k)_k$. That is for every $N \in \mathbb{N}$ there exists a $k > N$ such that

$$\epsilon_1, \ldots \epsilon_N = {\epsilon^k}_1, \ldots {\epsilon^k}_N.$$

But then $\pi(\underline{\epsilon}) \in W_k \subset W_N$ and thus $\pi(\underline{\epsilon}) \in W_N$ for every N. That is $\pi(\underline{\epsilon}) = z$, because J_f is a Cantor set and thus $\operatorname{diam}(W_N) \to 0$ as $N \to \infty$. $\square$

Proposition 1.7. *If $V_{\epsilon^k} = V_{\widehat{\epsilon}^k}$ and $\epsilon_1 \neq \widehat{\epsilon}_1$. Then the component W of $f^{-k}(U)$ with $\phi(V_{\epsilon^k}) = W \backslash J_f$ contains at least one critical point.*

Moreover if $\pi(\underline{\epsilon}) = \pi(\underline{\widehat{\epsilon}}) = z$ and and $\epsilon_1 \neq \widehat{\epsilon}_1$. Then z is a critical point for f.

Proof. Let $W' = f(W)$ then $\phi(V_{\sigma(\epsilon^k)}) = \phi(V_{\sigma(\widehat{\epsilon}^k)}) = W'$ and thus $V_{\sigma(\epsilon^k)} = V_{\sigma(\widehat{\epsilon}^k)}$ by property 3. of Proposition 1.4. Let x be any point of the later set, then $\phi(g_{\epsilon_1})$ and $\phi(g_{\widehat{\epsilon}_1})$ are two distinct preimages in W of the point $\phi(x) \in W'$. Hence the degree of the restriction $f : W \longrightarrow W'$ is at least 2 and thus W contains at least one critical point by the Riemann-Hurwitz formula. This proves the first statement of the Lemma. The second is an immediate consequence of Proposition 1.4 and the first statement:

$$\pi(\underline{\epsilon}) = \pi(\underline{\widehat{\epsilon}})$$

$$\Updownarrow$$

$$\bigcap_{k=1}^{\infty} \overline{\phi(V_{\epsilon^k})} = \bigcap_{k=1}^{\infty} \overline{\phi(V_{\widehat{\epsilon}^k})}$$

$$\Updownarrow$$

$$\forall k \in \mathbb{N} : \phi(V_{\epsilon^k}) = \phi(V_{\widehat{\epsilon}^k})$$

$$\Updownarrow$$

$$\forall k \in \mathbb{N} : V_{\epsilon^k} = V_{\widehat{\epsilon}^k}$$

Thus if $\epsilon_1 \neq \widehat{\epsilon}_1$ and $W_k \backslash J_f = \phi(V_{\epsilon^k})$. Then each W_k contains a critical point, $W_{k+1} \subset\subset W_k$ for all k and

$$z = \pi(\underline{\epsilon}) = \cap_{k=1}^{\infty} \overline{\phi(V_{\epsilon^k})} = \cap_{k=1}^{\infty} W_k.$$

Hence z is a critical point. $\square$

Corollary 1.8. *Let $z \in J_f$ be any point whose orbit $(f^n(z))_{n \geq 0}$ does not contain a critical point. Then*

$$\#\pi^{-1}(z) = 1.$$

Proof. Suppose $\pi(\underline{\epsilon}) = \pi(\underline{\widehat{\epsilon}}) = z$. We shall show that $\underline{\epsilon} = \underline{\widehat{\epsilon}}$. As a start $\epsilon_1 = \widehat{\epsilon}_1$ by Proposition 1.7. The Corollary now follows by induction since by the conjugacy property of π

$$\pi(\sigma^n(\underline{\epsilon})) = \pi(\sigma^n(\underline{\widehat{\epsilon}})) = f^n(z)$$

and by assumption this point is not critical, so that $\epsilon_n = \widehat{\epsilon}_n$ for all n by Proposition 1.7. $\square$

To shorten notation let us write $d_z = \deg(f, z)$ for any $z \in U'$.

Proposition 1.9. *For any $z \in J_f$*

$$\#\pi^{-1}(z) = d_z \#\pi^{-1}(f(z)).$$

More precisely there are d_z distinct numbers $j_1, \ldots j_{d_z} \in \{0, \ldots, d-1\}$ depending only on z such that $\pi(\underline{\epsilon}) = z$ if and only if $\pi(\sigma(\underline{\epsilon})) = f(z)$ and

$$\epsilon_1 \in \{j_1, \ldots, j_{d_z}\}.$$

Proof. Given $z \in J_f$ let W_k denote the connected component of $f^{-k}(U)$ containing z and let W'_k denote the connected component of $f^{-(k-1)}(U)$ containing $f(z)$. Then the degree of the restrictions $f : W_k \longrightarrow W'_{k-1}$ equals d_z for k sufficiently large, because z is the only point in the nested intersection of the W_k. Fix any such k_0, let $k \geq k_0$ and let V'_{k-1} denote the connected component of $\phi^{-1}(W'_{k-1} \backslash J_f)$ such that $V'_{k-1} = V_{\epsilon^{k-1}}$ for any ϵ^{k-1} with $\phi(V_{\epsilon^{k-1}}) = W'_{k-1} \backslash J_f$. Let $j_1, \ldots j_{d_z} \in \{0, \ldots, d-1\}$ be the d_z values of j for which $\phi(g_j(V'_{k-1})) = W_k \setminus J_f$ as provided by Proposition 1.2. Then the index set $\{j_1, \ldots j_{d_z}\}$ does not depend on the value of $k \geq k_0$ by nestedness of the sets V'_{k-1}. Hence $\pi(\underline{\epsilon}) = z$ if and only if $\epsilon_1 \in \{j_1, \ldots j_{d_z}\}$ and $\pi(\sigma(\underline{\epsilon})) = f(z)$. $\square$

Proof. (of Theorem 1.1) By the above Propositions π is a continuous and surjective semiconjugacy. In particular

$$\forall z \in J_f : \#\pi^{-1}(z) \geq 1.$$

Since no critical point of f is periodic and there are finitely many critical points counted with multiplicity, the total branching $\chi(z)$ along the orbit of an arbitrary point is uniformly bounded. In particular for any $z \in J_f$ there exists $N \in \mathbb{N}$ such that the orbit of $f^N(z)$ does not contain any critical point. Thus by Proposition 1.7

$$\#(\pi^{-1}(f^N(z))) = 1.$$

Finally we have $\chi(z) = d_z \cdot d_{f(z)} \cdot \ldots \cdot d_{f^{N-1}(z)}$, so that

$$\forall z \in J_f : \#\pi^{-1}(z) = \chi(z).$$

by induction on Proposition 1.9. $\square$

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